Research and Innovation Center of Excellence

Industrial Control
Part VII: Step-Response Analysis

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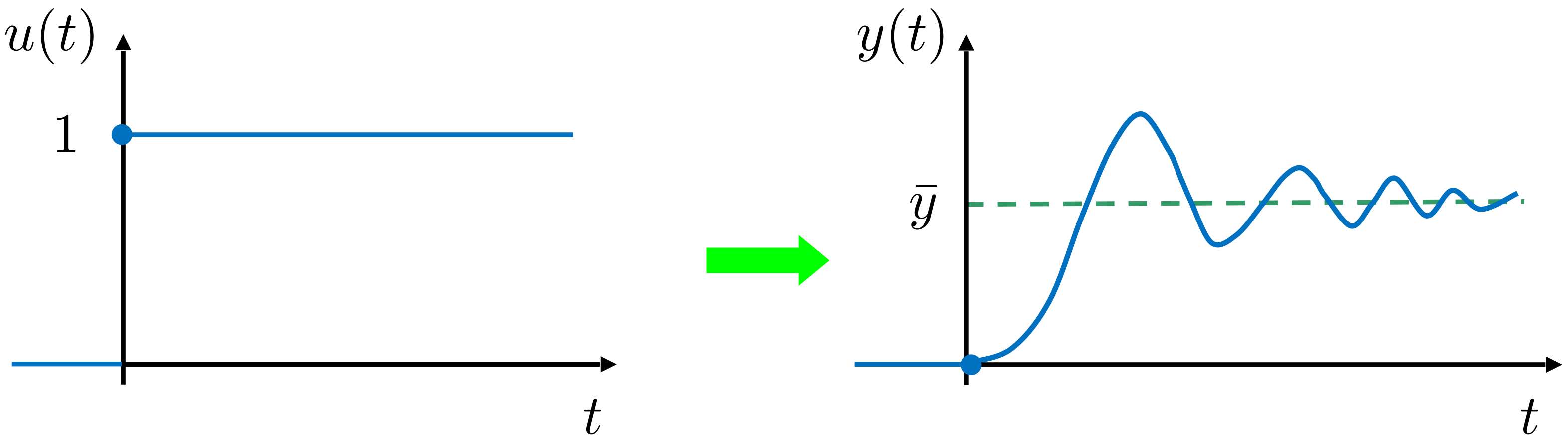
Step Response

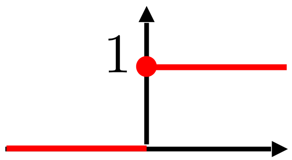
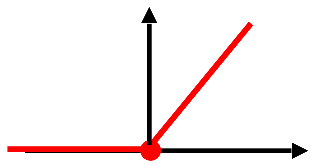
$$x(0) = 0\,; \quad u(t) = 1(t)$$



Recall that, for asymptotically stable systems, the step response describes the way the systems ”moves” from an equilibrium to another

Step Response (contd.)

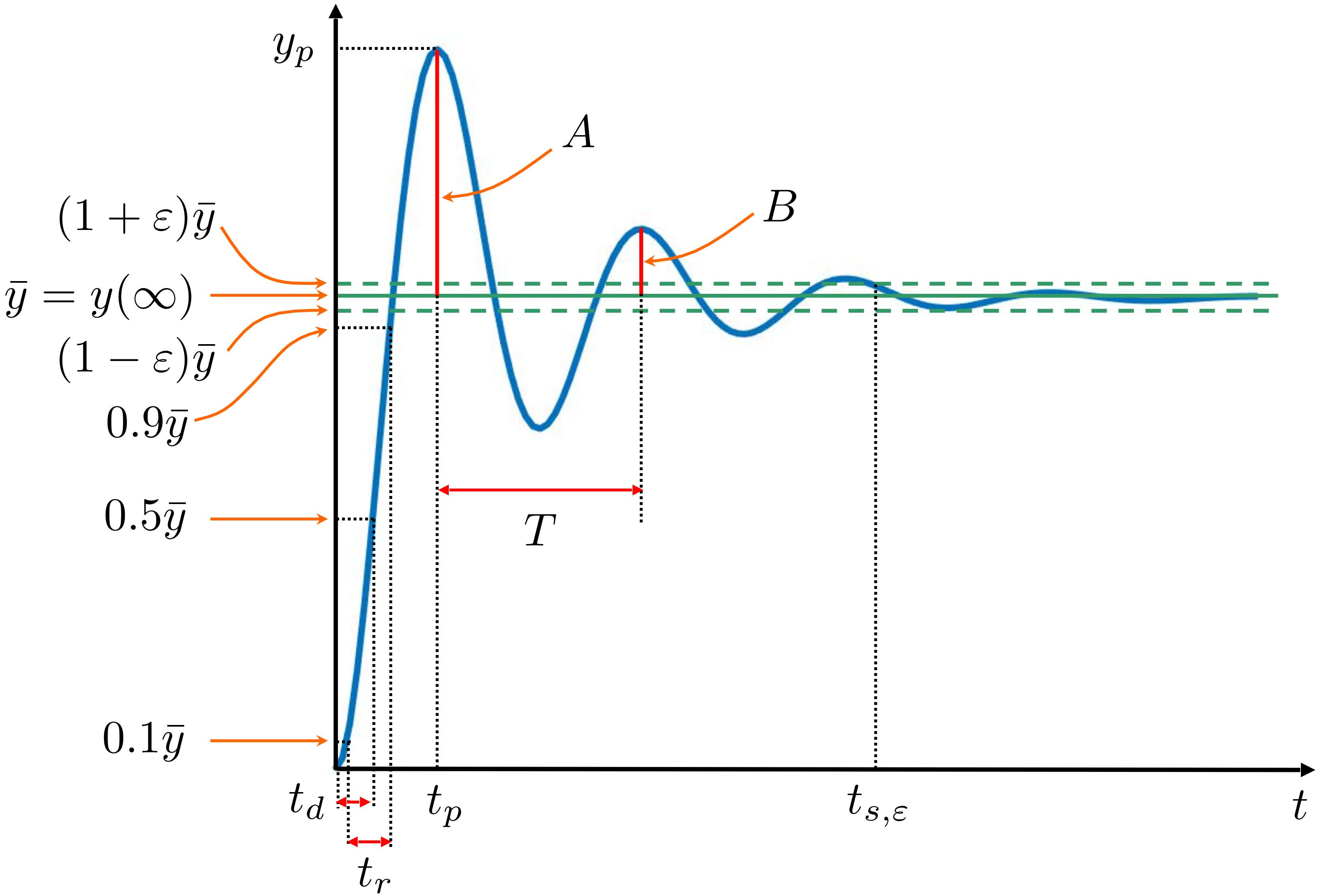
The knowledge of the step response allows to easily determine the response to other inputs related to the step function by linear transformations

time		
	$1(t)$	$t \cdot 1(t)$
Laplace	$\frac{1}{s}$	$\frac{1}{s^2}$

$$t \cdot 1(t) = \int_0^t 1(\tau) d\tau$$

→ The response to the ramp function is the integral of the step-response

Characteristic Parameters of the Step Response



Characteristic Parameters of the Step Response (contd.)

- Steady-state value: $\bar{y} = y(\infty)$
- Settling time: $t_{s,\varepsilon}$
- Rise time: t_r
- Delay time: t_d
- Peak time: t_p
- Peak value: y_p
- Maximum overshoot: $A = y_p - y(\infty)$
- Maximum percentage overshoot: $\Delta\% = 100 \cdot A/y(\infty)$
- "Period" of oscillations: T
- Damping factor: B/A

Step Response: First Order Systems

- **Case A)**

$$G(s) = \frac{\mu}{1+s\tau}; \quad \mu > 0; \; \tau > 0$$

strictly proper first-order system

↳ asymptotic stability

- **Case B)**

$$G(s) = \frac{\mu(1+sT)}{1+s\tau}; \quad \mu > 0; \; \tau > 0$$

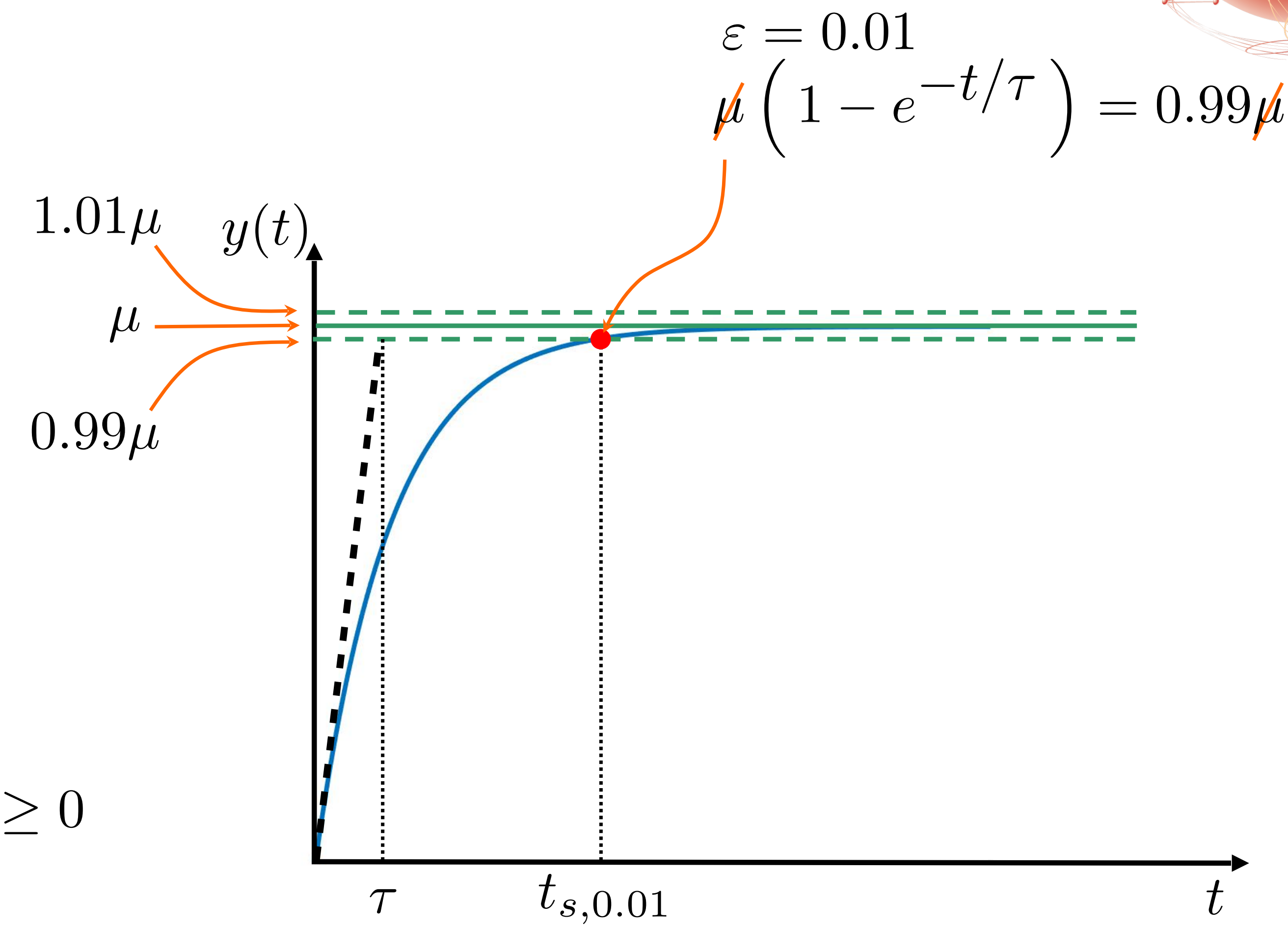
non strictly proper first-order system

↳ asymptotic stability

Step Response: First Order Systems (contd.)

- **Case A)**

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] \\
&= \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{\mu}{s} - \frac{\mu\tau}{1+s\tau}\right] \\
&= \mu\left(1-e^{-t/\tau}\right), \quad t \geq 0
\end{aligned}
$$



Settling-Time Calculation

For example, the settling time for $\varepsilon = 0.01$ can be characterised as follows:

$$1 - e^{-t/\tau} = 0.99 \rightarrow e^{-t/\tau} = 0.01 \rightarrow e^{t/\tau} = 100$$

$$t_{s,0.01} = \tau \ln 100 \simeq 4.6\tau$$

The calculation of the rising time t_r and the delay time t_d follows similar lines.

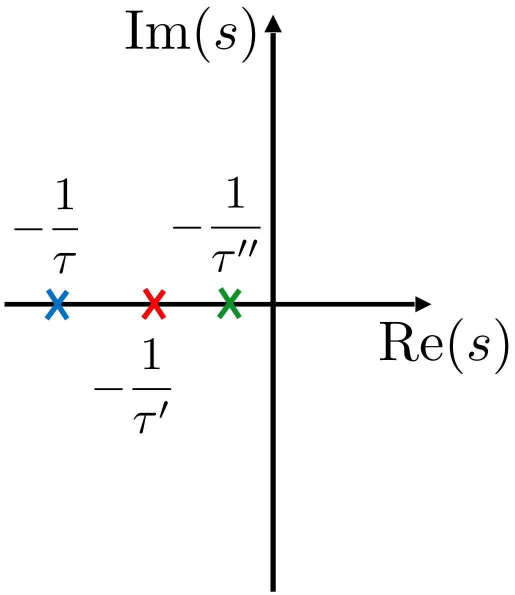
$t_{s,0.01}$

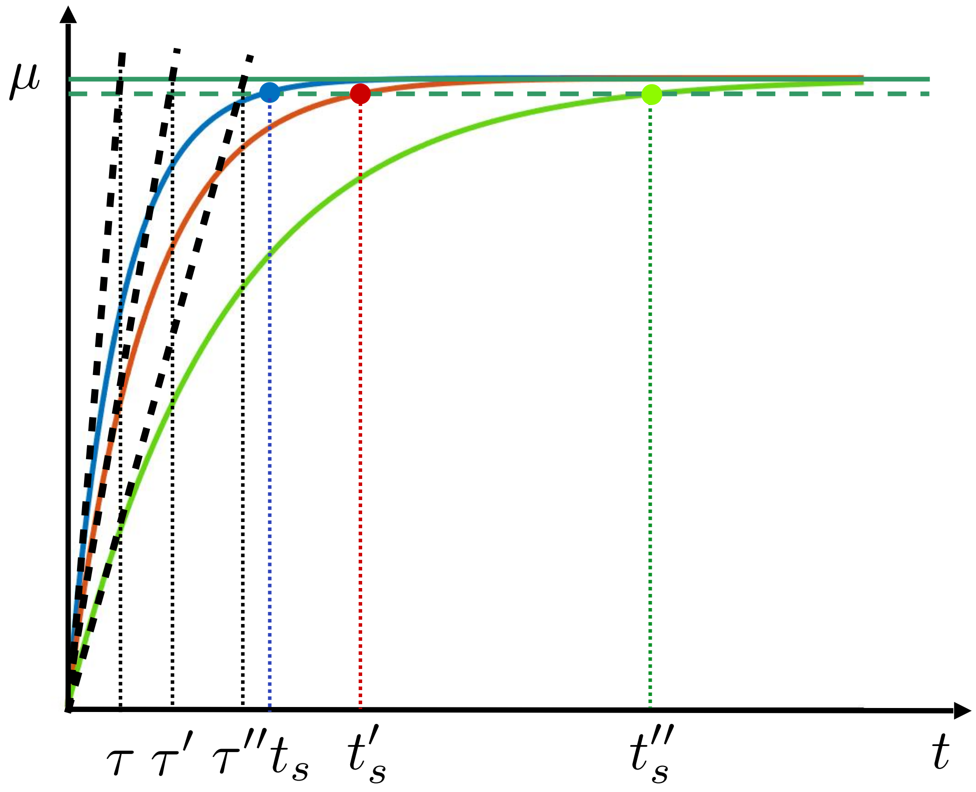
The following approximations are useful:

$$t_r \simeq 2.2\tau \qquad t_d \simeq 0.7\tau \qquad t_{s,0.05} \simeq 3\tau \qquad t_{s,0.01} \simeq 4.6\tau$$

Remark: without loss of generality, from now on we shall use t_s as a shorthand for $t_{s,0.01}$

Qualitative Analysis of the Step Response





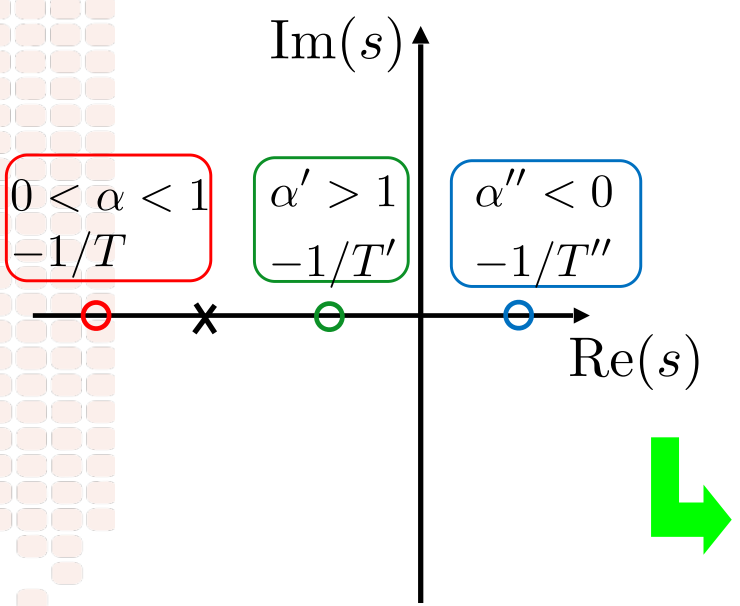
Step Response: First Order Systems (contd.)

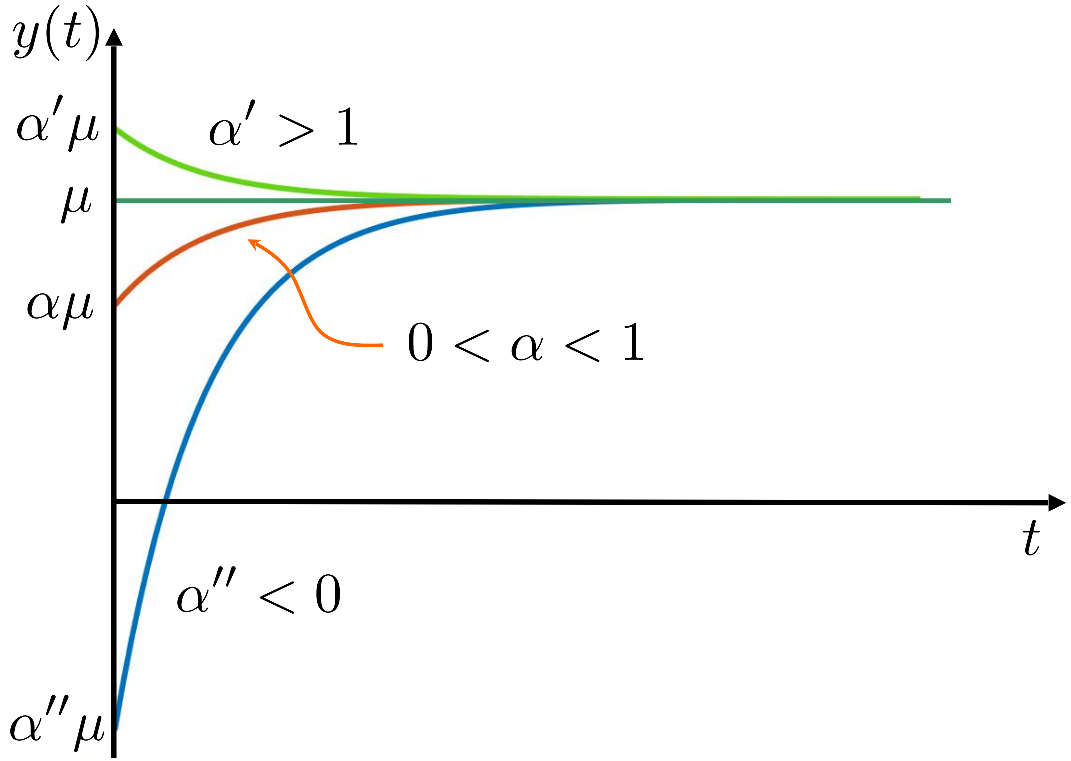
- **Case B)**

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] \\ &= \mathcal{L}^{-1}\left[\frac{\mu(1+sT)}{s(1+s\tau)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{\mu}{s}+\frac{\mu(T-\tau)}{1+s\tau}\right] \\ &= \mu\left(1+(\alpha-1)e^{-t/\tau}\right), \quad t \geq 0 \ \text{ with } \ T=\alpha\tau \end{aligned}$$

Note that (the system is not strictly proper): $\lim_{t\to 0^+} y(t) = \mu\frac{T}{\tau} \neq 0$

Qualitative Analysis of the Step Response





Step Response: Second Order Systems

- **Case A)**

$$G(s) = \frac{\mu}{(1+s\tau_1)(1+s\tau_2)}$$

real poles, no zeros

- **Case B)**

$$G(s) = \frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}$$

real poles, one zero

- **Case C)**

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$

complex poles, no zeros

- **Case D)**

$$G(s) = \frac{\varrho(1+sT)}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$

complex poles, one zero

Step Response: Second Order Systems (contd.)

- **Case A)**

$$G(s) = \frac{\mu}{(1 + s\tau_1)(1 + s\tau_2)}\,; \quad \mu > 0\,; \quad \tau_1 \neq \tau_2$$

$$\left.\begin{array}{l}\tau_1 > 0 \\ \tau_2 > 0\end{array}\right\}$$ asymptotic stability

Without loss of generality, assume $\tau_1 > \tau_2$

Step Response: Second Order Systems (contd.)

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau_1)(1+s\tau_2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{1+s\tau_1} + \frac{C}{1+s\tau_2}\right]$$

where

$$A = \left.\frac{\mu}{(1+s\tau_1)(1+s\tau_2)}\right|_{s=0} = \mu$$

$$B = \left.\frac{\mu}{s(1+s\tau_2)}\right|_{s=-1/\tau_1} = \frac{\mu}{-\frac{1}{\tau_1}(1-\frac{\tau_2}{\tau_1})} = \frac{\mu\tau_1^2}{\tau_2-\tau_1}$$

$$C = \left.\frac{\mu}{s(1+s\tau_1)}\right|_{s=-1/\tau_2} = \frac{\mu}{-\frac{1}{\tau_2}(1-\frac{\tau_1}{\tau_2})} = \frac{\mu\tau_2^2}{\tau_1-\tau_2}$$

Step Response: Second Order Systems (contd.)

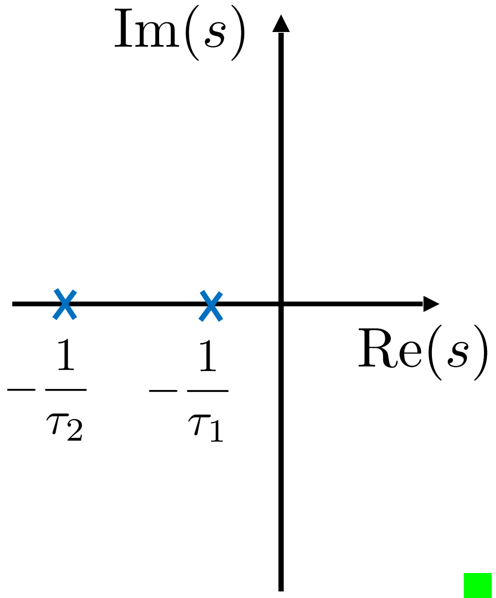
Hence:

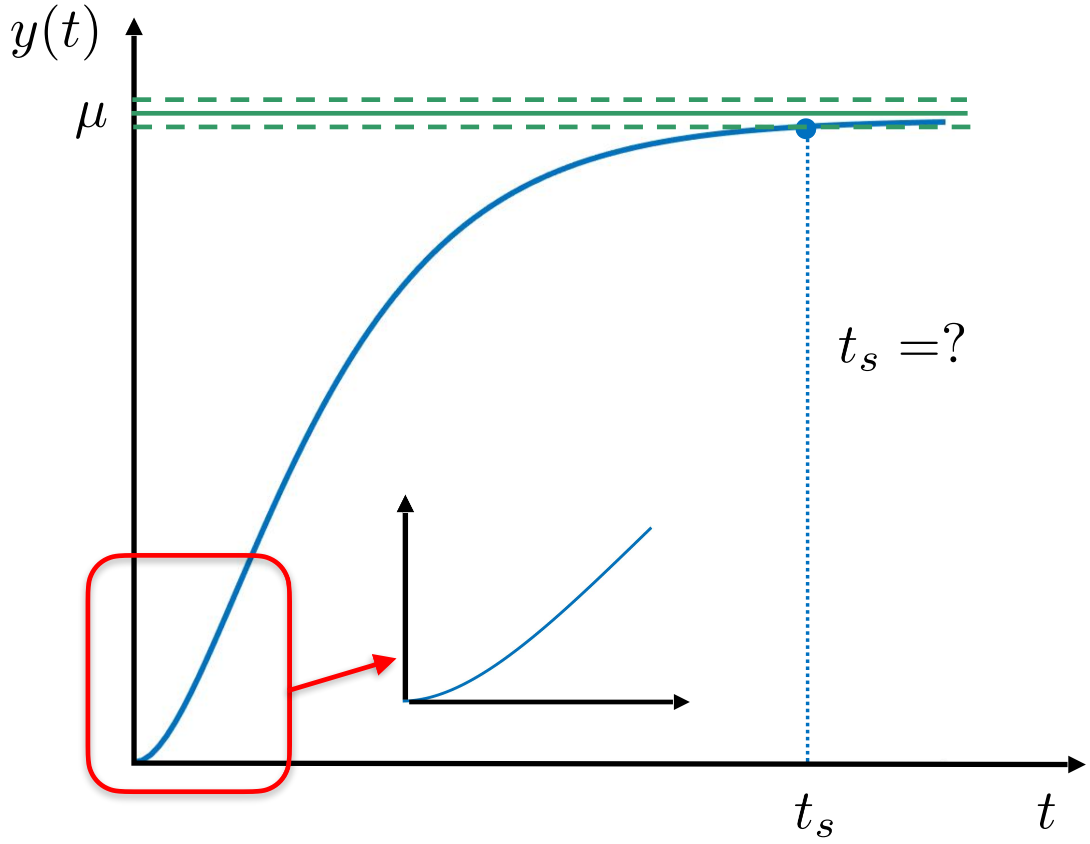
$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{\mu}{s} - \frac{\frac{\mu\tau_1^2}{\tau_2-\tau_1}}{1+s\tau_1} + \frac{\frac{\mu\tau_2^2}{\tau_1-\tau_2}}{1+s\tau_2}\right] \\
&= \mu\left(1 - \frac{\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1} + \frac{\tau_2}{\tau_1-\tau_2}e^{-t/\tau_2}\right), \quad t \geq 0
\end{aligned}
$$

Characteristics:

- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = 0$
- $\ddot{y}(0) = \dfrac{\mu}{\tau_1\tau_2} > 0$

Qualitative Analysis of the Step Response





Approximate Calculation of the Settling Time

If $\tau_1 \gg \tau_2$:

$$y(t) = \mu \left(1 - \frac{\tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_2}{\tau_1 - \tau_2} e^{-t/\tau_2} \right), \quad t \geq 0$$

$$\simeq \mu \left(1 - e^{-t/\tau_1} \right), \quad t \geq 0$$

$$t_s \simeq 4.6\tau_1$$

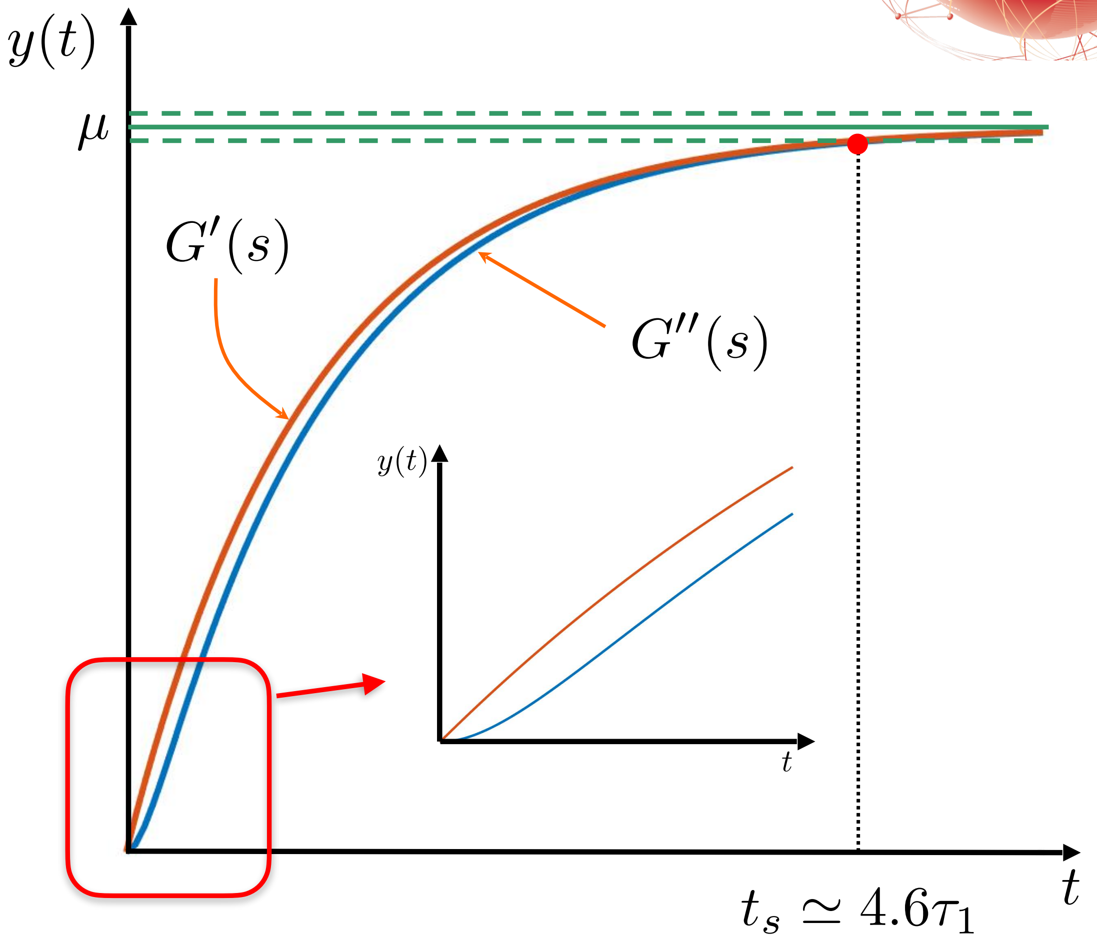
In general, in the absence of zeros, the most influential poles on the qualitative behaviour of the step response are the ones **closer to the imaginary axis**.

Qualitative Analysis: Comparison Between First and Second Order

$$G'(s) = \frac{\mu}{1 + s\tau_1}$$

$$G''(s) = \frac{\mu}{(1 + s\tau_1)(1 + s\tau_2)}; \ \tau_1 \gg \tau_2$$

- The main difference lies in the initial transient behaviour
- For a given settling time, the step-response in the second-order case without zeros has a "slower" dynamics



Step Response: Second Order Systems (contd.)

- **Case B)**

$$G(s) = \frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}\,; \quad \mu > 0\,; \quad \tau_1 \neq \tau_2$$

$$\left.\begin{array}{l} \tau_1 > 0 \\ \tau_2 > 0 \end{array}\right\} \Longrightarrow \text{asymptotic stability}$$

Without loss of generality, assume $\tau_1 > \tau_2$

Step Response: Second Order Systems (contd.)

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau_1)(1+s\tau_2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{1+s\tau_1} + \frac{C}{1+s\tau_2}\right]$$

where

$$A = \left.\frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}\right|_{s=0} = \mu$$

$$B = \left.\frac{\mu(1+sT)}{s(1+s\tau_2)}\right|_{s=-1/\tau_1} = \frac{\mu(1-T/\tau_1)}{-\frac{1}{\tau_1}(1-\frac{\tau_2}{\tau_1})} = \frac{\mu\tau_1(\tau_1-T)}{\tau_2-\tau_1}$$

$$C = \left.\frac{\mu(1+sT)}{s(1+s\tau_1)}\right|_{s=-1/\tau_2} = \frac{\mu(1-T/\tau_2)}{-\frac{1}{\tau_2}(1-\frac{\tau_1}{\tau_2})} = \frac{\mu\tau_2(\tau_2-T)}{\tau_1-\tau_2}$$

Step Response: Second Order Systems (contd.)

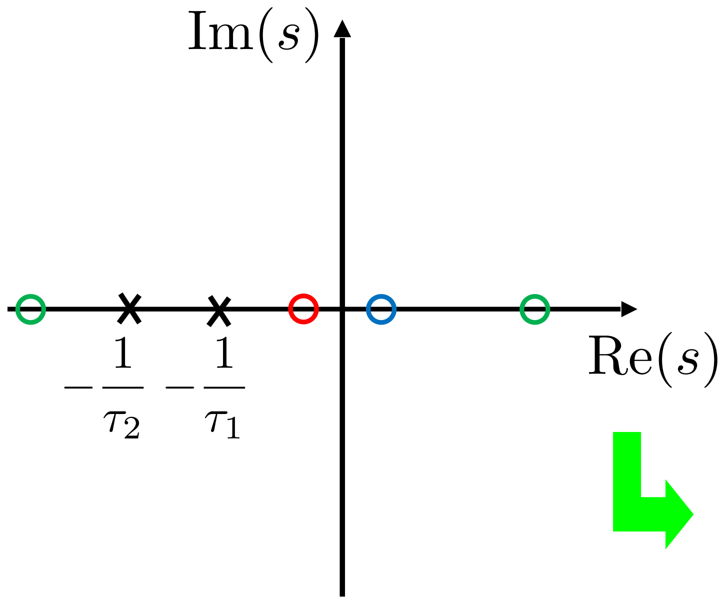
Hence:

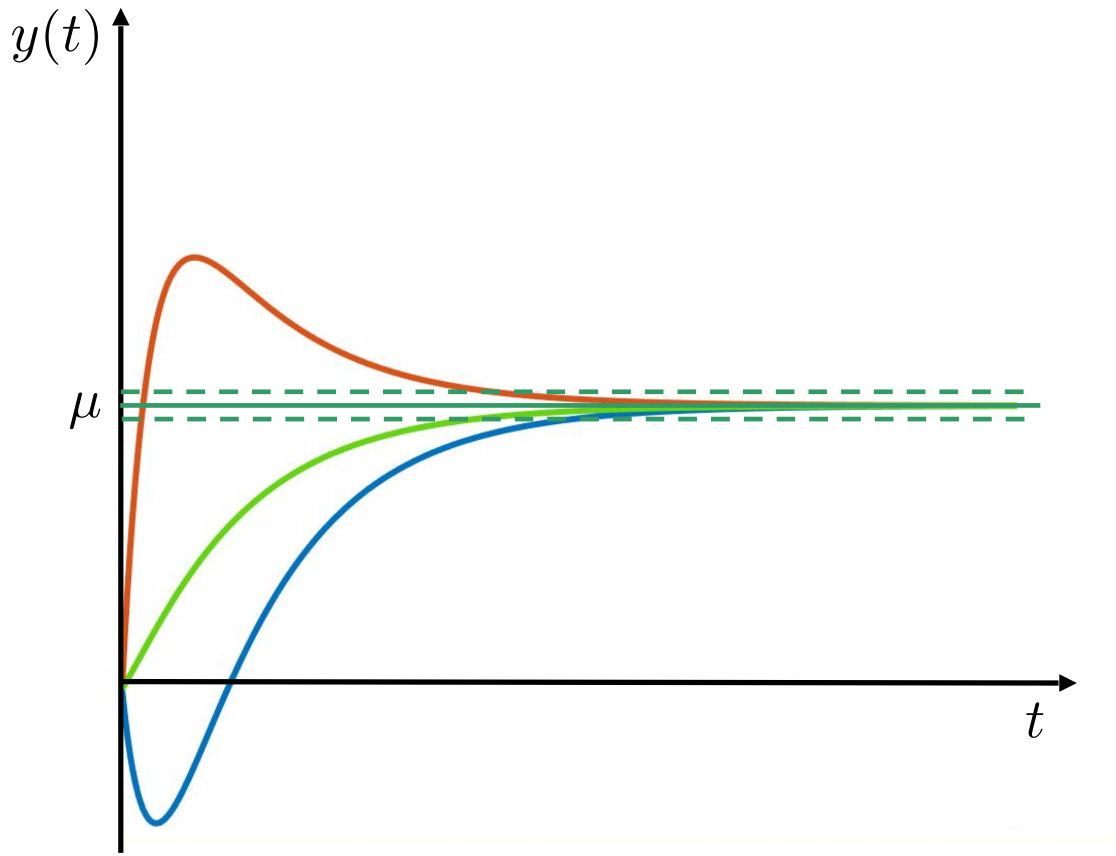
$$y(t) = \mu \left(1 - \frac{\tau_1 - T}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_2 - T}{\tau_1 - \tau_2} e^{-t/\tau_2} \right), \quad t \geq 0$$

Characteristics:

- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = \dfrac{\mu T}{\tau_1 \tau_2} \begin{cases} > 0, & \text{if } T > 0 \\ < 0, & \text{if } T < 0 \end{cases}$

Qualitative Analysis of the Step Response





- zero with little influence (green)
- overshoot (red)
- undershoot (blue)

Step Response: Second Order Systems (contd.)

- **Case C)**

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$

$$\mu = G(0) = \frac{\varrho}{\sigma^2+\omega^2}$$

poles: $-\sigma \pm j\omega$

$\sigma > 0$ → asymptotic stability

$\omega > 0$

$\varrho > 0$

Step Response: Second Order Systems (contd.)

$$Y(s) = \frac{G(s)}{s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\sigma s + \sigma^2 + \omega^2}$$

$$As^2 + 2A\sigma s + A\sigma^2 + A\omega^2 + Bs^2 + Cs = \varrho$$

$$\begin{cases} A + B = 0 \\ 2A\sigma + C = 0 \\ A(\sigma^2 + \omega^2) = \varrho \end{cases} \rightarrow \begin{cases} A = \dfrac{\varrho}{\sigma^2 + \omega^2} = \mu \\ B = -\mu \\ C = -2\sigma\mu \end{cases}$$

$$Y(s) = \mu \left[\frac{1}{s} - \frac{s + 2\sigma}{s^2 + 2\sigma s + \sigma^2 + \omega^2} \right] = \mu \left[\frac{1}{s} - \frac{s + \sigma + \sigma}{(s + \sigma)^2 + \omega^2} \right]$$

$$= \mu \left[\frac{1}{s} - \frac{s + \sigma}{(s + \sigma)^2 + \omega^2} - \frac{\sigma}{\omega} \frac{\omega}{(s + \sigma)^2 + \omega^2} \right]$$

Step Response: Second Order Systems (contd.)

Hence: $$y(t) = \mu\left[1 - e^{-\sigma t}\cos(\omega t) - \frac{\sigma}{\omega}e^{-\sigma t}\sin(\omega t)\right], \quad t \geq 0$$

$$= \mu\left[1 - e^{-\sigma t}\left(\cos(\omega t) + \frac{\sigma}{\omega}\sin(\omega t)\right)\right], \quad t \geq 0$$

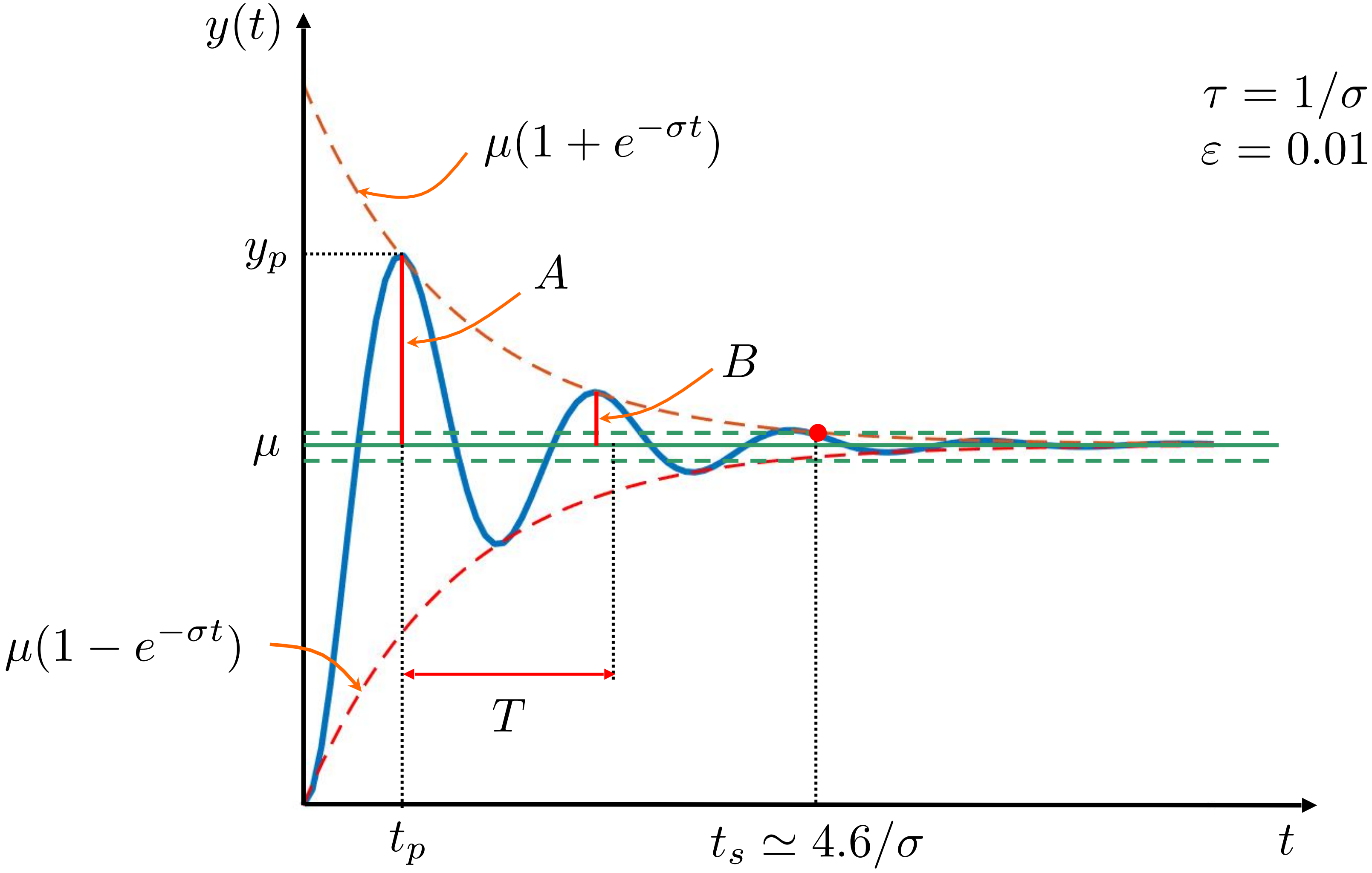
$$= \mu \underbrace{\left[1 - \frac{\sqrt{\sigma^2+\omega^2}}{\omega}e^{-\sigma t}\sin(\omega t + \varphi)\right]}_{\text{damped oscillations}}, \quad t \geq 0$$

$$\text{where } \varphi = \arccos\left(\frac{\sigma}{\sqrt{\sigma^2+\omega^2}}\right)$$

Characteristics:

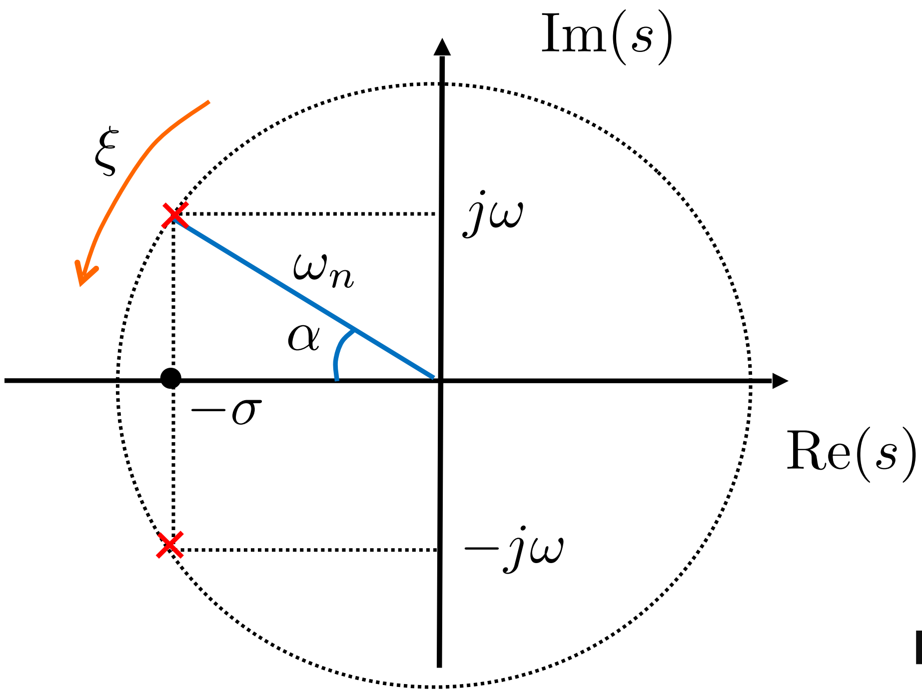
- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = 0$
- $\ddot{y}(0) = \varrho > 0$

Qualitative Analysis of the Step Response



Characteristic Parameters of the Step Response

Recall from Part 4:



$$\omega_n^2 = \sigma^2 + \omega^2$$

$$\omega_n \xi = \sigma$$

$$\omega_n \sqrt{1 - \xi^2} = \omega$$

$$G(s) = \frac{\varrho}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Parameters:

ω_n natural angular frequency:

$\xi = \cos(\alpha)$ damping ratio

Characteristic Parameters of the Step Response (contd.)

and:

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)} = \frac{\varrho}{(s+\sigma)^2+\omega^2}$$

$$= \frac{\varrho}{s^2 + \underbrace{2\sigma}_{2\xi\omega_n} s + \underbrace{\sigma^2+\omega^2}_{\omega_n^2}} = \frac{\varrho}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$G(s) = \frac{\varrho/\omega_n^2}{1+\frac{2\xi}{\omega_n}s+\frac{1}{\omega_n^2}s^2} = \frac{\mu}{1+\frac{2\xi}{\omega_n}s+\frac{1}{\omega_n^2}s^2}$$

where: $\mu := \frac{\varrho}{\omega_n^2}$

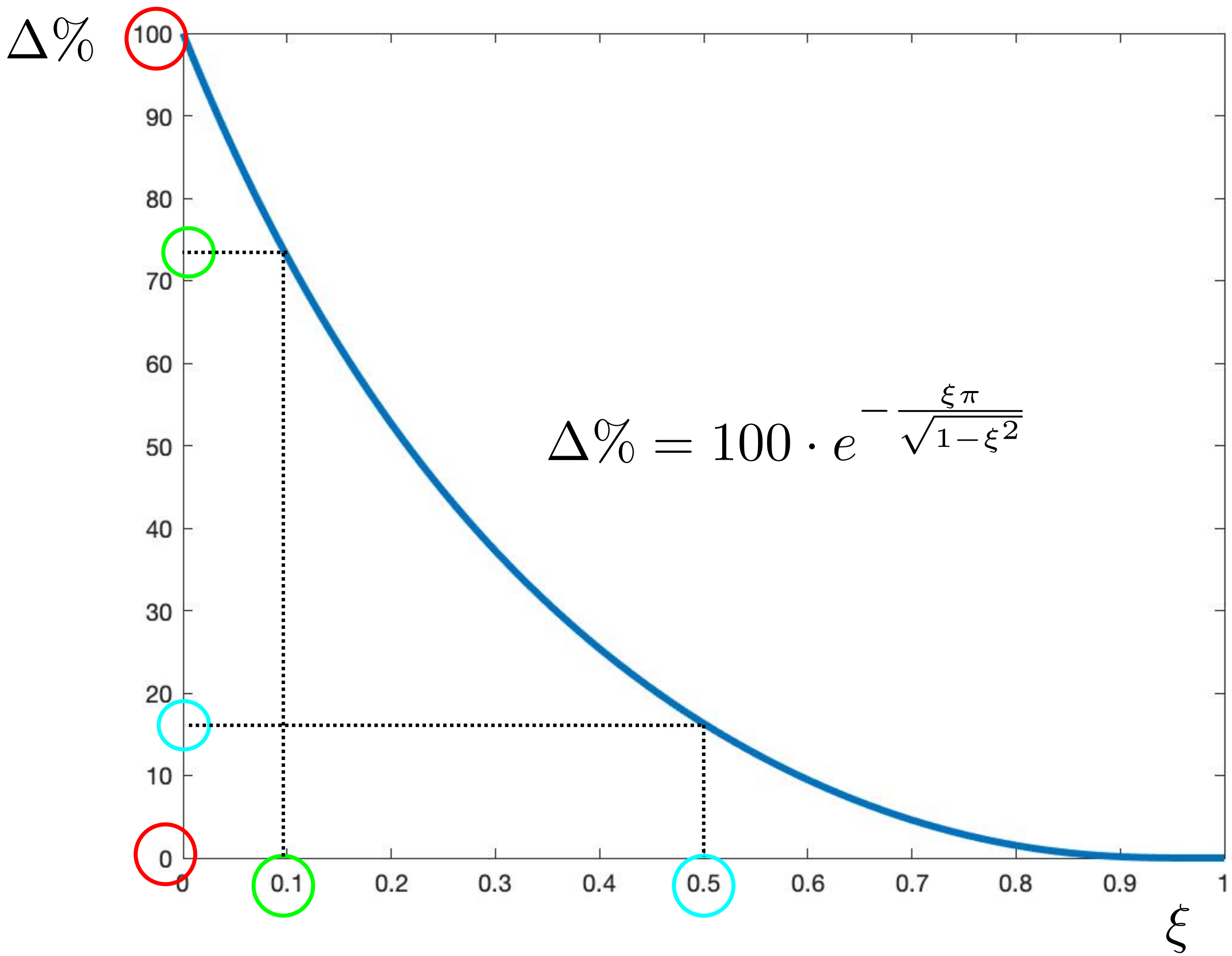
Characteristic Parameters of the Step Response (contd.)

Hence:

- Settling time: $t_s \simeq \frac{5}{\sigma} = \frac{4.6}{\xi\omega_n}$
- Peak time: $t_p = \frac{\pi}{\omega} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$
- Peak value: $y_p = \mu\left[1 + e^{-\frac{\sigma\pi}{\omega}}\right] = \mu\left[1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}\right]$
- Maximum percentage overshoot: $\Delta\% = 100 \cdot \frac{A}{\mu} = e^{-\sigma\pi/\omega} = 100 \cdot e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$
- "Period" of oscillations: $T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$
- Damping factor: $\frac{B}{A} = \cdots = \Delta^2 = e^{-2\sigma\pi/\omega} = e^{-\frac{2\xi\pi}{\sqrt{1-\xi^2}}}$

only depend on ξ
but **not** on ω_n

Maximum Percentage Overshoot



$$\Delta\% = 100 \cdot e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

Limit Cases

- **No damping:** $\xi = 0$

$$G(s) = \frac{\varrho}{s^2 + \omega_n^2} \qquad \text{poles: } \pm j\omega_n$$

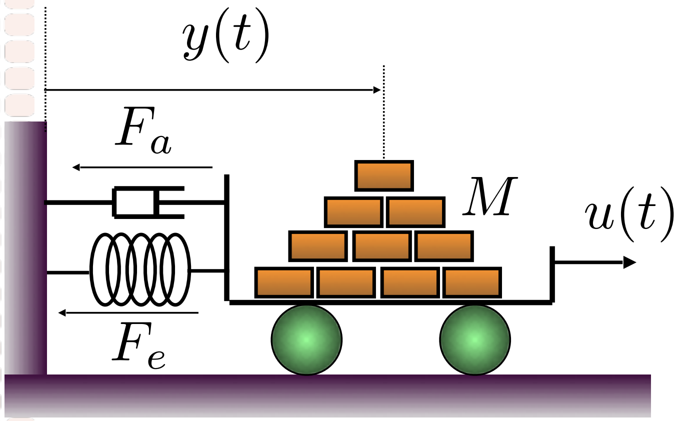
Undamped oscillations

- **Full damping:** $\xi = 1$

$$G(s) = \frac{\varrho}{(s + \omega_n)^2} \qquad \text{poles: } -\omega_n;\ -\omega_n$$

No oscillations at all

Example 1



$$G(s) = \frac{1}{Ms^2 + hs + k} = \frac{1/M}{s^2 + \frac{h}{M}s + \frac{k}{M}}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \frac{1/M}{s^2 + \frac{h}{M}s + \frac{k}{M}}$$

Hence:

$$\mu = G(0) = \frac{1}{k}$$

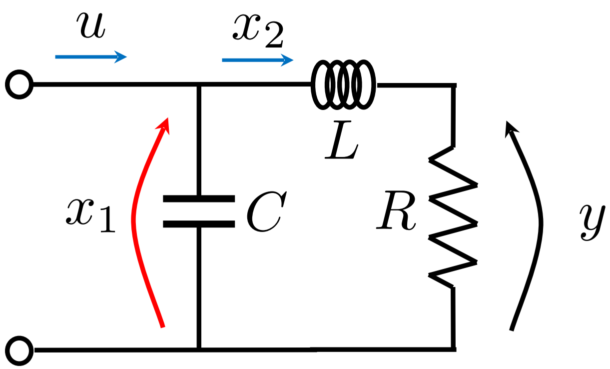
$$2\xi\omega_n = \frac{h}{M}$$

$$\omega_n^2 = \frac{k}{M}$$

$$\omega_n = \sqrt{\frac{k}{M}}$$

$$\xi = \frac{h}{2\sqrt{kM}}$$

Example 2



$$\begin{cases} C\dot{x}_1 = u - x_2 \\ L\dot{x}_2 = x_1 - Rx_2 \\ y = R\,x_2 \end{cases}$$

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \qquad B = \begin{bmatrix} 1/C \\ 0 \end{bmatrix} \qquad C = [0 \;\; R]$$

$$G(s) = [0 \;\; R] \begin{bmatrix} s & 1/C \\ -1/L & s + R/L \end{bmatrix}^{-1} \begin{bmatrix} 1/C \\ 0 \end{bmatrix} = \cdots = \frac{R/(LC)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\omega_n = \frac{1}{\sqrt{LC}}; \quad \xi = \frac{R}{2}\sqrt{\frac{C}{L}}; \quad \mu = R$$

Step Response: Second Order Systems (contd.)

- **Case D)**

$$G(s) = \frac{\mu(1+sT)}{1+\dfrac{2\xi}{\omega_n}s+\dfrac{1}{\omega_n^2}s^2}\,; \quad 0<\xi<1\,;\ \omega_n>0\,;\ \mu>0$$

Characteristics of the step response:

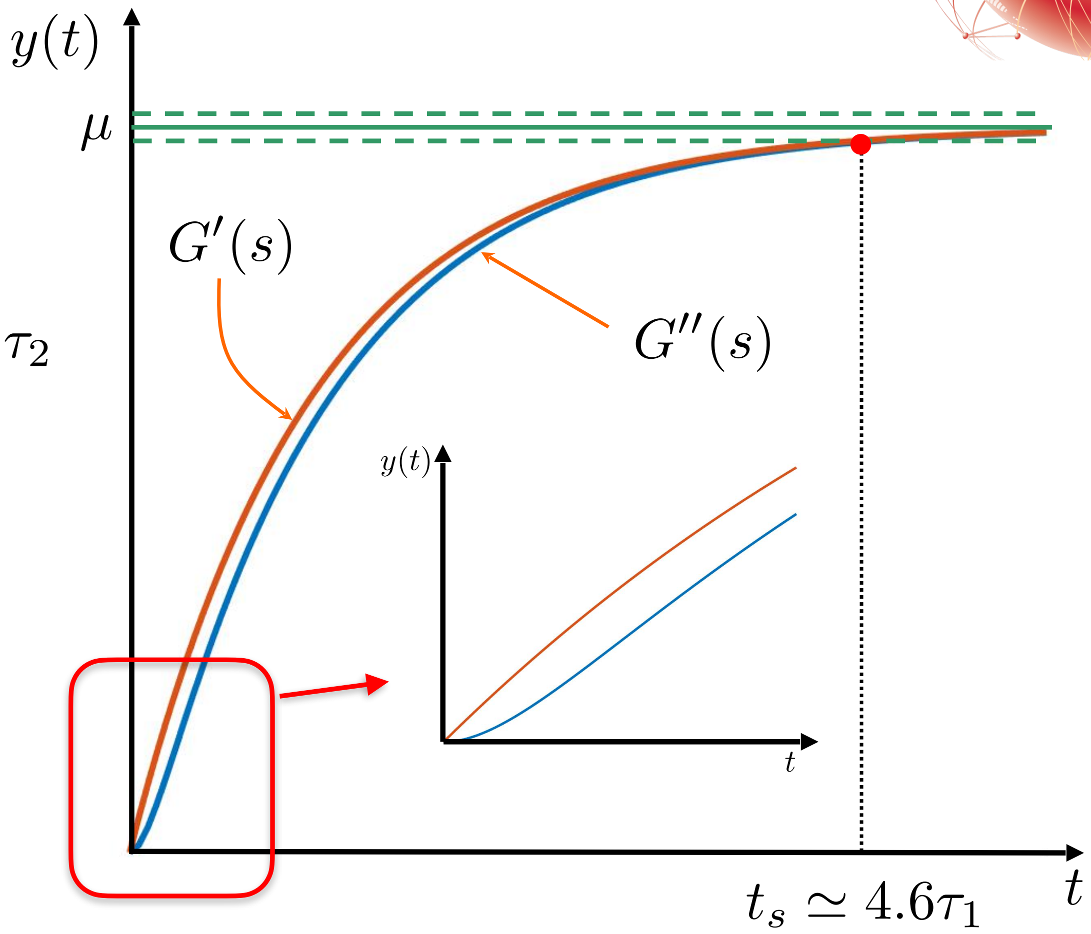
- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = \mu T \omega_n^2 \begin{cases} >0, & \text{if } T>0 \\ <0, & \text{if } T<0 \end{cases}$

Qualitative Analysis: Comparison between first and second order

$$G'(s) = \frac{\mu}{1 + s\tau_1}$$

$$G''(s) = \frac{\mu}{(1 + s\tau_1)(1 + s\tau_2)}; \ \tau_1 \gg \tau_2$$

- The main difference lies in the init transient behaviour
- For a given settling time, the step response in the second-order cas without zeros has a "slower" dynamics

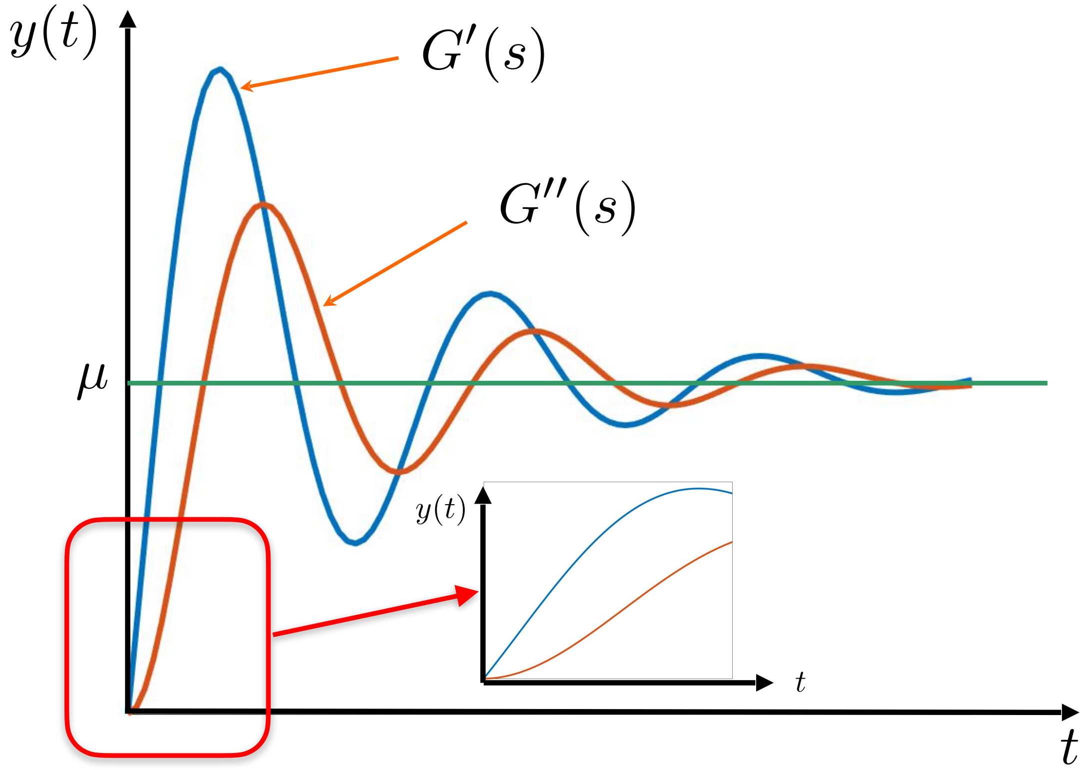


Qualitative Analysis:
Comparison between Case C) (no zeros) and Case D) (one zero)

$$G'(s) = \frac{\mu}{1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

$$G''(s) = \frac{\mu(1 + sT)}{1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

- Again, the main difference lies in the initial transient behaviour
- For a given settling time, the step-response in Case C) without zeros has a "slower" dynamics



Step Response for Systems of Order > 2

For simplicity, consider the case of real poles only:

$$G(s) = \frac{\mu}{s^g} \frac{\prod_{i=1}^{m}(1+sT_i)}{\prod_{i=1}^{n}(1+s\tau_i)}$$

Recall (in the absence of common factors in $G(s)$):

Asymptotic Stability $\Longleftrightarrow$ $\mathrm{Re}(\text{poles}) < 0$, $g < 0$

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]$$

Step Response for Systems of Order > 2 (contd.)

- **Initial Value Theorem**

$$\lim_{t \to 0^+} y(t) = \lim_{s \to \infty} \cancel{s} \frac{1}{\cancel{s}} G(s) \begin{cases} = 0\,, & \text{if } m < n \\ \neq 0\,, & \text{if } m = n \end{cases}$$

- **Final Value Theorem**

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} G(s) \begin{cases} = \mu\,, & \text{if } g = 0 \\ = 0\,, & \text{if } g < 0 \end{cases}$$

Dominant Poles Approximation

Again, for simplicity, consider the case of real poles:

$$Y(s) = G(s)\frac{1}{s} = \frac{\alpha_0}{s} + \frac{\alpha_1}{1+s\tau_1} + \cdots + \frac{\alpha_n}{1+s\tau_n}$$

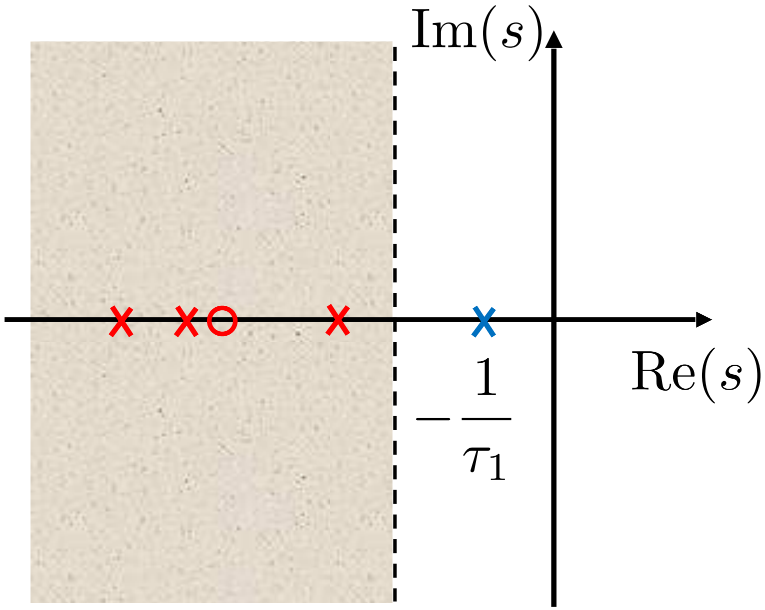
$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] \\ &= \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1} + \cdots + \frac{\alpha_n}{\tau_n}e^{-t/\tau_n} \end{aligned}$$

Assuming: $\tau_1 > \tau_2 > \cdots > \tau_n$

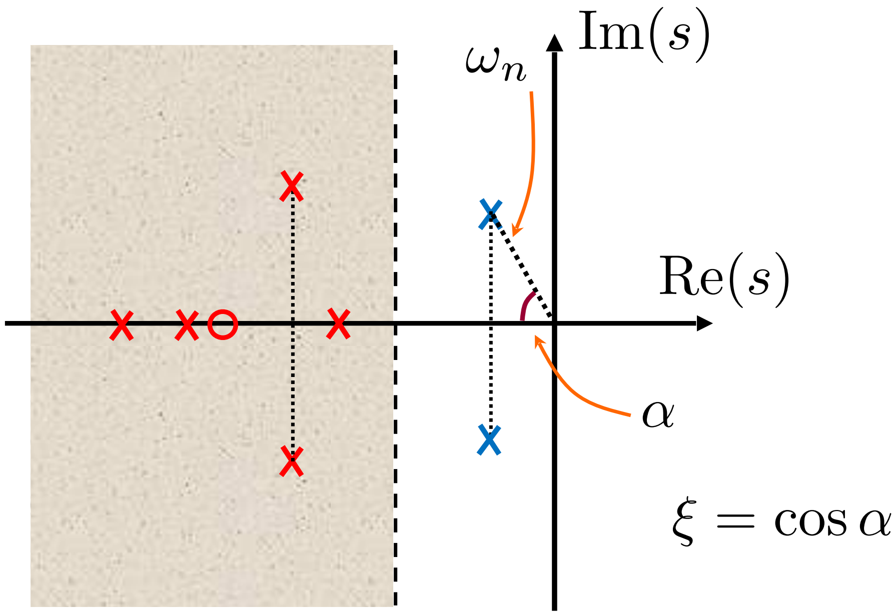
$$\begin{aligned} y(t) &= \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1} + \cdots + \frac{\alpha_n}{\tau_n}e^{-t/\tau_n} \\ &\simeq \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1} \end{aligned}$$

dominant component, hence: $t_s \simeq 5\tau_1$

Dominant Poles Approximation: Real Poles



Dominant Poles Approximation: Real and Complex Poles



Example

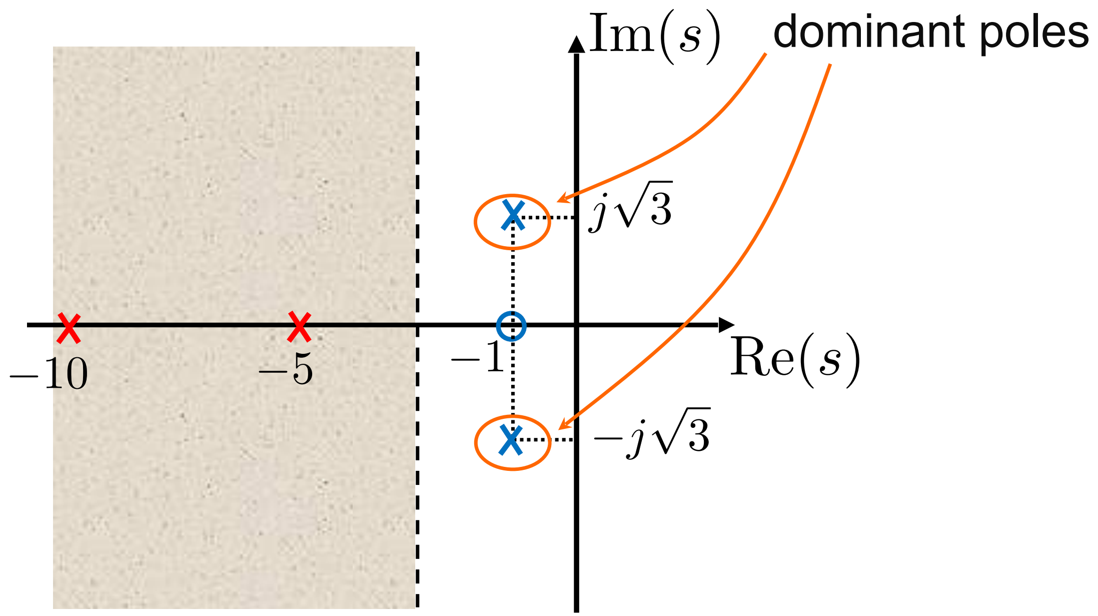
$$G(s) = \frac{400(1+s)}{(1+0.2s)(1+0.1s)(s^2+2s+4)}$$

$\omega_n = 2$
$\xi = 1/2$

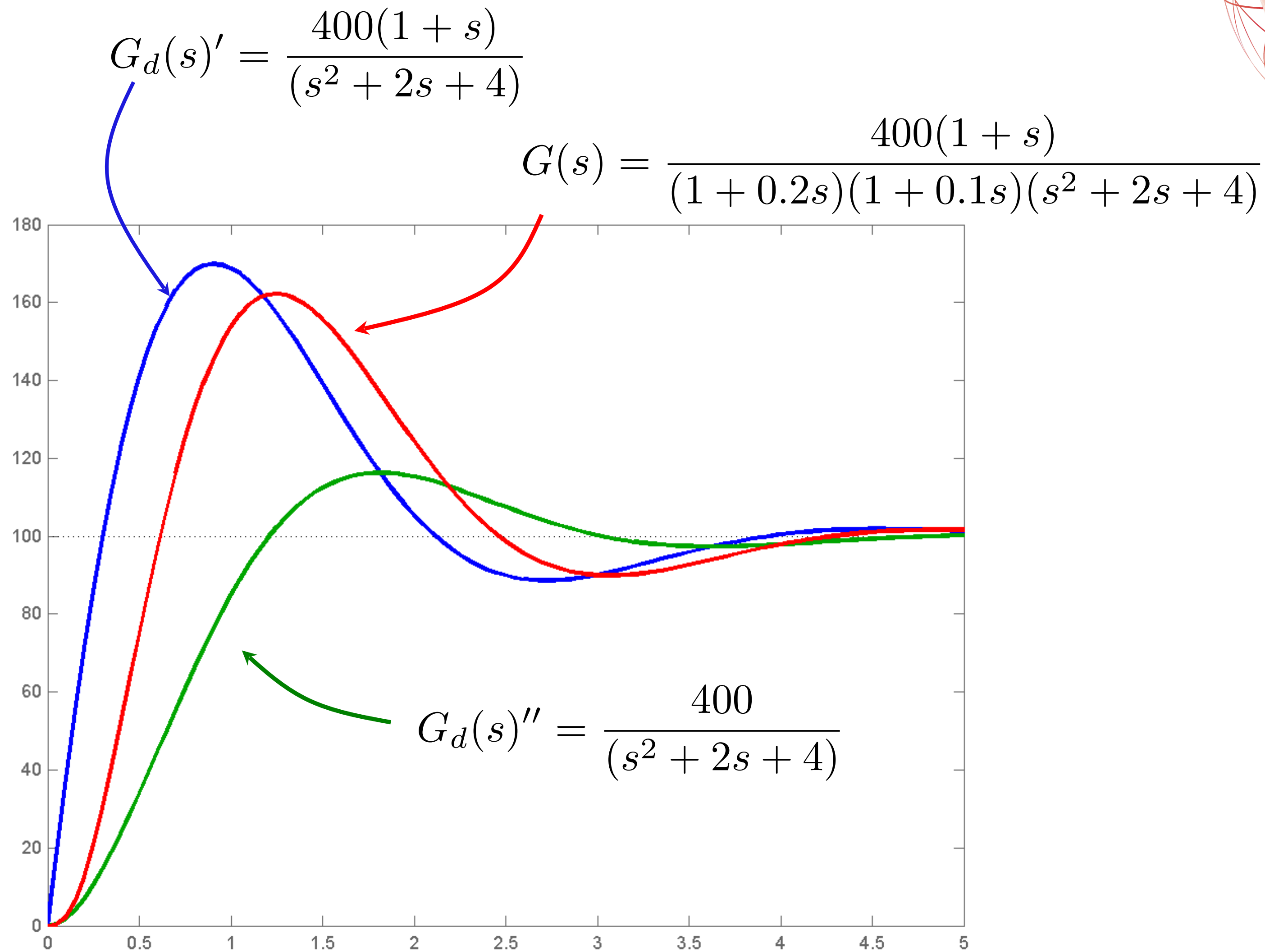
$\mu = G(0) = 100$

poles: -5
-10
$-1 \pm j\sqrt{3}$

zero: -1



Example (contd.)



Equivalent Pole Approximation

Suppose that all poles are real and negative:

$$G(s) = \frac{\mu}{(1+s\tau_1)(1+s\tau_2)\cdots(1+s\tau_n)}\,; \qquad \tau_i > 0,\; i = 1,\ldots,n$$

$$G_e(s) = \frac{\mu}{1+s\tau_e}\,, \qquad \tau_e := \sum_{i=1}^{n} \tau_i$$

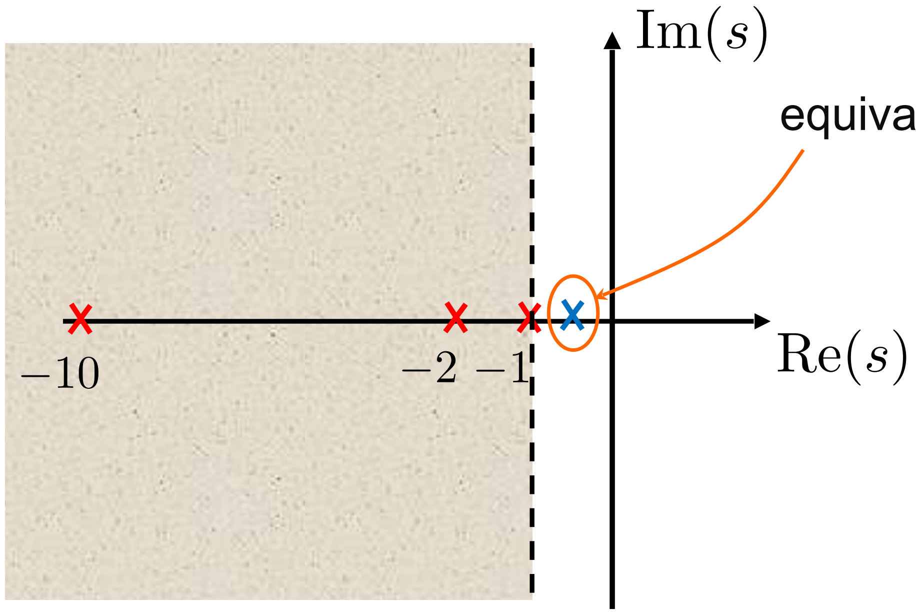
dominant component, hence: $t_s \simeq 5\tau_e$

Example

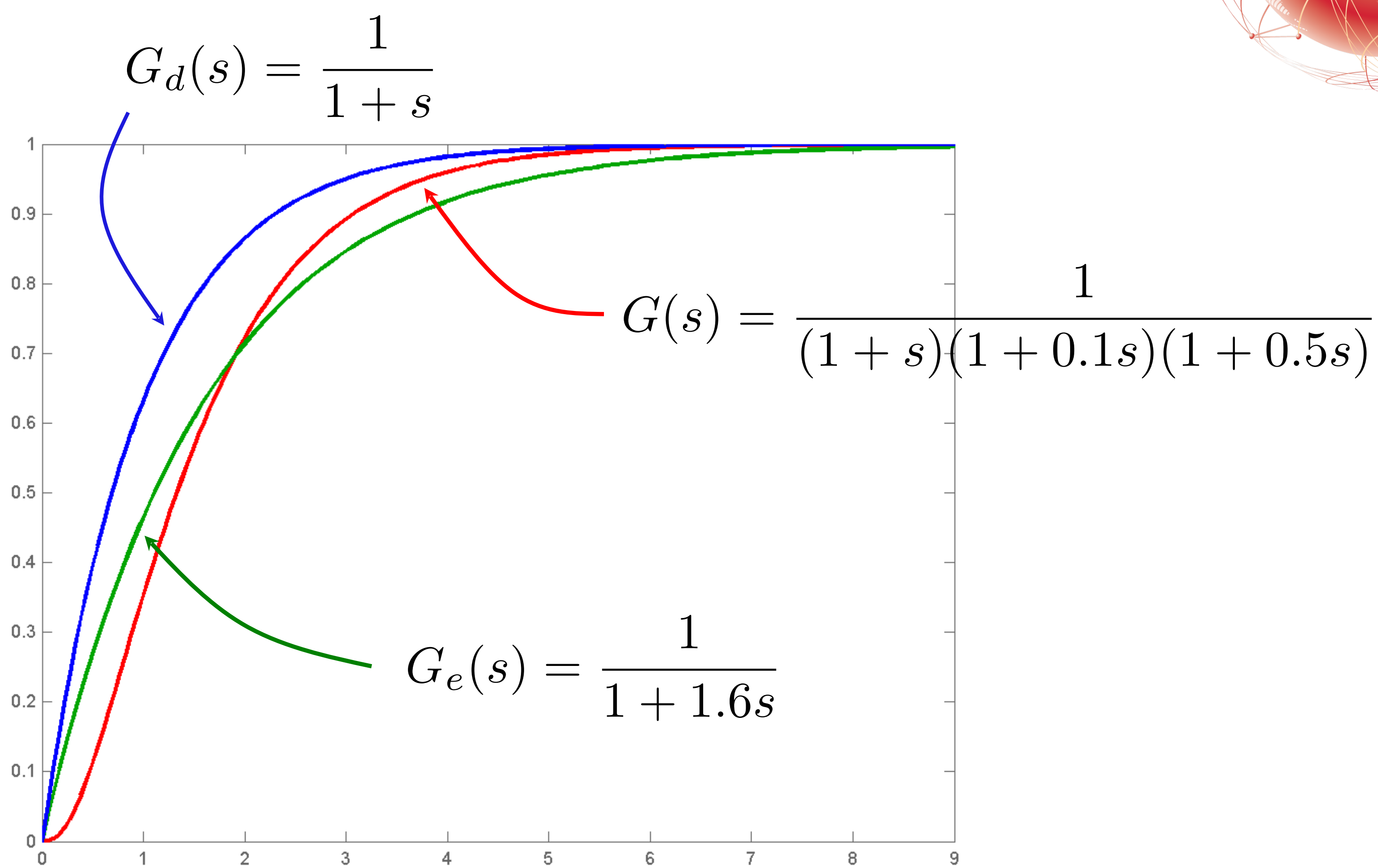
$$G(s) = \frac{1}{(1+s)(1+0.1s)(1+0.5s)}$$

$\mu = G(0) = 1$

poles: -1
-2
-10



Example (contd.)



Dominant and Equivalent Poles Approximation: Remarks

- When using the **dominant poles** approximation:
  - It is important to “preserve” the gain
  - Zeros located close to the imaginary axis have to be properly taken into account
- The **equivalent pole** approximation can only be used when all poles are real and negative
- Both approximations are useful in qualitative analysis and the for initial controller’s design steps